MSE 101: Mathematics: Sam Cooper and David Dye

Data Analysis Question Sheet

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Exercises

Logs and curve sketching

1. Simplify the following expressions:

   (a) $5e^{\ln(A)}$
   (b) $7\ln(\mathrm{e}^{-A^2})$
   (c) $3e^{\ln(5)+A}$
   (d) $\log_{10}(0.1 \times 10^A)$
   (e) $(\mathrm{e}^{(2\ln(A))})^2$
   (f) $2^{2\log_2(A)}$

2. Solve the following for x:

   (a) $7^x = 13$
   (b) $3^x = \mathrm{e}^{x+2}$
   (c) $3\mathrm{e}^x = 7\mathrm{e}^{2x}$
   (d) $9 = 5 \times 2^x$
   (e) $7^{2x} - 9 = 0$
   (f) $3\log_3(3^{3x}) = 3$

3. Using the change of base rules, rewrite the following expressions in the natural logarithm (*i.e.* base e).

   (a) $\log_2(7)$
   (b) $\log_\pi(7)$
   (c) $\log_{2e}(7)$
   (d) $\log_3(x) - \log_5(x)$

4. Plot the functions $y = \mathrm{e}^x$ and $y = \mathrm{e}^{x/2}$ on the same linear axes. Now take the natural log (*i.e.* ln) of each of these functions and plot them on a second graph.

5. One the same axes, sketch the function $f(x) = -x^2 + 5x - 6$ as well as the functions $f(x+2)$, $f(x)+2$ and $f(x)/2$ (ideally in different colours!)

6. Sketch the following functions, labelling all roots, asymptotes, stationary points and inflection points. Also, state the domain and range of these functions:

   (a) $y = x^2 - 3x - 28$
   (b) $y = 4x^3 - 8x^2 - 11x - 3$
   (c) $y = \frac{2x-7}{x^2-2}$
   (d) $y = \frac{x^2-2}{2x-7}$
   (e) $y = 3\mathrm{e}^{2x} - 2$
   (f) $y = \ln(x\mathrm{e}^x)$

Root finding

7. Use the bisection method to find the roots for the following functions to 2 s.f.:

   (a) $y = 3^x - 5 + x$ starting from the interval $[1, 2]$
   (b) $y = 3x^x - 7x - 2$ starting from the intervals $[0, 1]$ and $[2, 3]$

8. Use the Newton-Raphson method to find the roots of the following functions 3 s.f.:

   (a) $y = x^5 - x^2 - 4$ starting from $x_0 = 0.5$ and then $x_0 = -1.2$ (write down the number of iterations required for each).
   (b) $y = x^4 - e^x$ finding all three roots.

Power series

9. Using the Maclaurin series to re-express the following functions (find the first 4 terms):

(a) $f(x) = \sin(2x)$ (c) $f(x) = x^2 - e^{2x}$
(b) $f(x) = 2\cos(x)$ (d) $f(x) = \frac{1}{x-3}$

10. Using the Taylor series to re-express the following functions (find the first 4 terms):
    (a) $f(x) = \sinh(x)$ around the point $x = 1$.
    (b) $f(x) = 2\cos^2(x)$ around the point $x = \pi$.
    (c) $f(x) = x + e^{2x}$ around the point $x = \mathrm{e}$.
11. For the function $f(x) = \mathrm{e}^{2x}$, find the Maclaurin series up to the term containing x^3. On the same axes sketch the function and its 0^{th}, 1^{st}, 2^{nd}, and 3^{rd} order approximations.
12. Find the first four terms of the Maclaurin series for the function $f(x) = \cos(x)$.
13. Find the first four terms of the Maclaurin series for the function $y = \sin^2(x) + \cos(x)$.
14. Try to find the first term of the Maclaurin series for the function $y = \ln(x)$...
15. Find the first four terms of the Taylor series of the function $y = \ln(x)$, expanded around the point $x = 1$.

Gaussians

16. Data recorded from the random variable x has been modelled using a Normal distribution with a mean of 3 and a standard deviation of 7.
    (a) What is the probability of being more than 20% away from the mean?
    (b) What is the probability of being less than 3% away from the mean?
    (c) What is the probability of being between in the interval $-100 < x < 4$?
17. Data recorded from the random variable x has been modelled using a Normal distribute with a mean of 5 and a standard deviation of 2. How many terms of the Maclaurin series for the error function are required to calculate the probability of $x < 6$, accurate to better than 0.001% (use the approximation built into http://www.wolframalpha.com/ as the "true" answer).

Error Analysis

18. Evaluate the uncertainty $\sigma_f(a, b, c, ...)$ given the following data. Comment, for each case, on effect of the magnitude of the different sources of error on your answer.
    (a) $f = a^2$ where, $a = 25 \pm 1$
    (b) $f = a - 2b$ where, $a = 100 \pm 3, b = 45 \pm 2$
    (c) $f = \frac{a}{b}(c^2 + \sqrt{d})$ where, $a = 0.100 \pm 0.003, b = 1.00 \pm 0.05, c = 50.0 \pm 0.5, d = 100 \pm 8$
    (d) $f = 1 - \frac{1}{a}$ where, $a = 50 \pm 2$
19. The volume $V = xyz$ of a rectangular block is measured by determining the lengths of its sides x, y and z. From the scatter of the measurements a standard error of 0.01% is assigned to each dimension. What is the standard error in V if (a) the scatter is due to errors in reading the instrument or (b) if it is due to temperature fluctuations?

Problems

1. For the function $f(x) = 2\frac{x^2-1}{x^2-4}$

   (a) Sketch the curve, annotating the position of its stationary points, roots and asymptotes.

   (b) Give the domain and range of this function

   (c) Find the interval in which the Newton-Raphson approximation can be successfully used to find each of the roots.

2. Find the Taylor Series expansion of the function $f(x) = e^{2x} - \ln(x)$ around the point $x = 3$ up to the x^3 term. Use this truncated series to evaluated $f(2.8)$ and at $f(2)$. Each each case find the percentage difference between the true and approximate answers. Explain this finding.

3. The error function is useful for evaluating probabilities of normally distributed variables and is defined as

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \text{e}^{-t^2}\, dt$$

   It is not possible to evaluate this integral directly; however, a power series expansion can be used to approximate it. This first seven differentials, evaluated at zero, are given below.

   $f(0) = 0$
   $f'(0) = \frac{2}{\sqrt{\pi}}$
   $f''(0) = 0$
   $f^{(3)}(0) = -\frac{4}{\sqrt{\pi}}$
   $f^{(4)}(0) = 0$
   $f^{(5)}(0) = \frac{24}{\sqrt{\pi}}$
   $f^{(6)}(0) = 0$
   $f^{(7)}(0) = -\frac{240}{\sqrt{\pi}}$

   Using this information, construct a Maclaurin series representation of the error function by finding the values of α, β and γ:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{\alpha} + \frac{x^5}{\beta} - \frac{x^7}{\gamma} + ... \right)$$

   A bored PhD student has measured the length of all 4000 hairs on their right arm and decides to model them using a normal distribution. They found the average length to be 12 mm with a standard deviation of 3 mm. Using the following equation and your power series approximation for the error function found above, determine, how many hairs are 11 mm ± 5%.

$$P(X < x) = \frac{1}{2} \left[1 + \text{erf}\left(\frac{x - \mu}{\sigma\sqrt{2}} \right) \right]$$

4. A weight W is suspended from the centre of a steel bar which is supported at its ends, and the deflection at the centre y is measured using a dial gauge indicator. The following values are obtained:

W/kg	y/μm
0.0	1642
0.5	1483
1.0	1300
1.5	1140
2.0	948
2.5	781
3.0	590
3.5	426
4.0	263
4.5	77

(a) Plot the points on a graph and draw the best line by eye. Make a guess of the standard error in the slope using a transparent ruler and see what reasonable limits for the line might be.

(b) Compute the best value for the slope and intercept, and their standard error using least squares minimisation (Chapter 6), and compare to your estimate in (a).

5. The temperature around a weld in a thin sheet is hypothesised to vary according to

$$T = T_0 + \frac{q}{2\pi\kappa d} \exp\left\{-\frac{v}{2\alpha}x\right\} K_0 \left\{\frac{v}{2\alpha}r\right\}$$

where K_0 is a Bessel function of type K and zeroth order. Here T_0 is the starting temperature of the sheet, q is the heat input, κ the thermal conductivity, α the thermal diffusivity, d the sheet thickness and v the welding velocity. x is the distance from the heat source along the welding direction and $r^2 = x^2 + y^2$ the distance from the heat source, where y is the distance from the heat source perpendicular to the welding direction, see Figure.

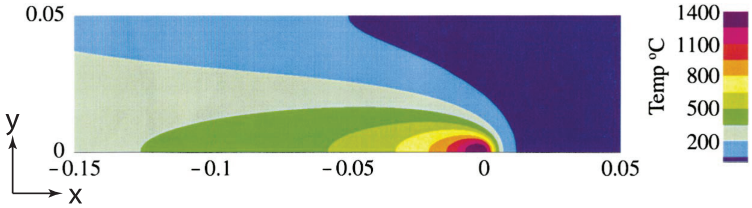
Temperature is measured using a thermocouple at a distance y from the weld centreline, with the aim of inferring the heat input q. Once the heat input is known, this can be used to, for example, model the thermal strains and therefore to make a finite element model for welding to predict welding-induced distortions and residual stresses.

q is difficult to measure directly because not all of the power applied to the welding torch Q ends up as heat in the plate - there are losses from the welding arc. So we assume that $q = \eta Q$, where η is the thermal efficiency - for TIG welding, this is on the order of 50%. y is also not easy to measure with sufficient precision, so it is left as a fitting parameter, but is estimated to be around 4 mm. In contrast, the material properties are well known and welding speed can be measured accurately.

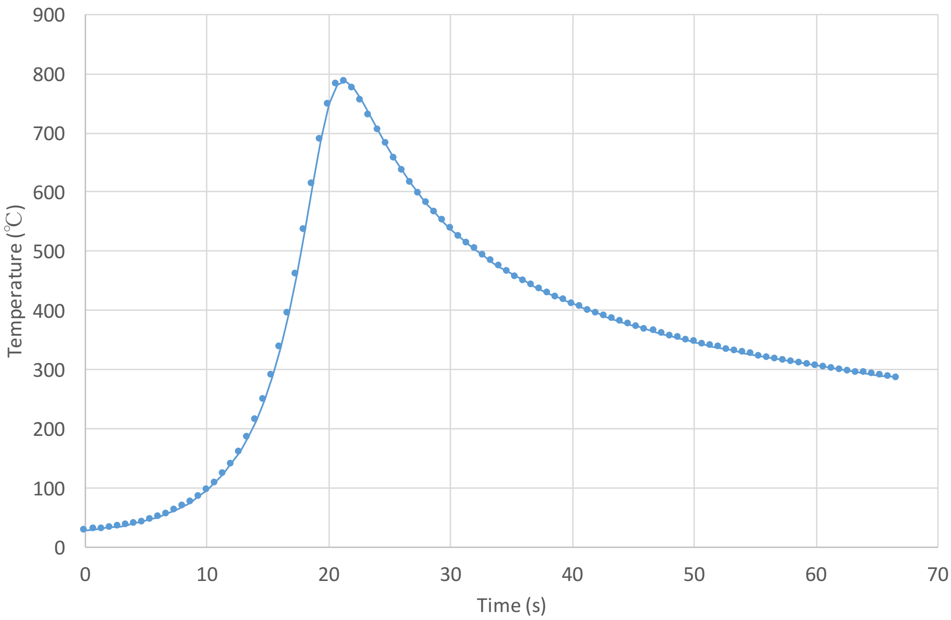
Therefore we can reformulate the equation in terms of time, by making the substitution $x = x_0 - vt$, to obtain

$$T = T_0 + \frac{\eta Q}{2\pi\kappa d} \exp\left\{-\frac{v}{2\alpha}(x_0 - vt)\right\} K_0 \left\{\frac{v}{2\alpha}\sqrt{(x_0 - vt)^2 + y^2}\right\}$$

For a particular weld in a mild steel, it is known that $T_0 = 20\,°\text{C}$, $\kappa = 46.2\,\text{Wm}^{-1}\text{K}^{-1}$, $\alpha = 11.7 \times 10^{-6}\,\text{m}^2\,\text{s}^{-1}$, $d = 2\,\text{mm}$, $Q = 600\,\text{W}$ and $v = 1.5\,\text{mm}\,\text{s}^{-1}$. y, η and x_0 are unknown and must be found. The data obtained in an experiment are plotted below, and are provided as a CSV file separately. Using Matlab, obtain a non-linear least squares estimate of y, η and x_0, and their associated uncertainties. In addition, plot the variation of χ^2 with y and η, for sensible values of these two fitting parameters.



Schematic of the temperature distribution around a welding torch (for a different problem).



Thermocouple data measured in the vicinity of a weld.